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FIXED POINT THEOREM FOR P-CONTRACTION MAPPING IN COMPLETE METRIC SPACE

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ABSTRACT

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The concept "weak topological contraction" and a generalization of Banach contraction mappings called "*p*-contraction" has been used to prove fixed point theorem for orbitally complete mapping from a metric space into itself satisfying metric *p*-contraction.

1. INTRODUCTION-

Banach [2] proved the following famous fixed point theorem:
Let X be a complete metric space with metric d and let f be a mapping from X into itself such that there exists $r \in [0,1)$ with $d(fx, fy) \leq r d(x,y)$ for every $x, y \in X$, then f has a unique fixed point.
Later this theorem is generalized in several directions. For example Ciric [2] proved a fixed point theorem for a quasi contraction. Sujuki and Takahasi [4] also proved a fixed point theorem for a weakly contractive mapping which is generalization of Banach contraction principle. Many authors has done the work in this field.

Recently Pathak H.K. [5] has give a result in fixed point theory by using *p*-contraction mapping, which is more general than the key ingredient of the Banach contraction principle. The aim of this paper is to extend the work of Pathak H.K.[5].

An operator $f : X \to X$ is said to be **orbitally continuous** if

$$f^{n_i}x \to u \text{ then}$$

$$f(f^{n_i}x) \to fu \text{ as } x \to \infty$$

A metric space (X, d) is said to be ***f*-orbitally complete** if f is a self mapping of X and if any Cauchy subsequence $\{f^{n_i}x\}$ in orbit $O(x, f)$ where $x \in X$, converges in X.

Let (X, d) be a metric space. A mapping

$$f : Y \subset X \to X$$

is said to be a metric p-contraction mapping if Y is f-invariant and it satisfies the following inequality

$$(f(x), f^2x) \leq p(x)d(x, fx) \text{ for all } x \text{ in } Y \text{ and}$$

$$p : Y \to [0,1]$$

Theorem 1: Let (X, d) be a metric space and f is orbitally complete. Suppose $f : X \to X$ be a p-contraction mapping such that,

$$d(fx, f^2x) \leq p(x)\{\tfrac{1}{2}[d(x, fy) + d(x, fx) + d(y, fy)]\} \qquad (1.1)$$

and $p : Y \to [0,1], Y \subset X$. Then f has a unique fixed point.

Proof. Let x be an arbitrary point of X construct an iterative sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_0 = x$, $x_n = fx_{n-1}$ for $n \in N$; Now using (1.1) we observe

$$d(x_n, x_{n+1}) = d(fx_{n-1}, f^2x_{n-1}) \qquad (1.2)$$

$$\leq p(x_{n-1})\left\{\frac{1}{2}\left[d(x_{n-1}, f^2x_{n-1}) + d(x_{n-1}, fx_{n-1}) + d(fx_{n-1}, f^2x_{n-1})\right]\right\}$$

$$\leq p(x_{n-1})\left\{\frac{1}{2}\left[d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right]\right\}$$

$$\leq p(x_{n-1})\left\{\frac{1}{2}\left[2d(x_n, x_{n-1})\right]\right\}$$

$$\leq p(x_{n-1})d(x_n, x_{n-1})$$

By using property of p-contraction we observe that

$$d(x_1, x_2) \leq p(x_0)d(x_0, x_1), d(x_2, x_3) \leq p(x_1)d(x_1, x_2)\ldots \qquad (1.3)$$

Continuing this process, we obtain

$$d(x_n, x_{n+1}) \leq \prod_{i=1}^{n} p_i d(x_0, x_1) \qquad (1.4)$$

where $p_i = p(x_{i-1}) = p(f^{i-1}(x_0)), i \in N$. Since

$$\max\{p(x_0), \sup_{x \in X} p(f(x))\} \leq \lambda < 1. \qquad (1.5)$$

It follows from (1.4)

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1) \quad \forall n \in N \qquad (1.6)$$

For $m > n(m, n \in N)$, we have

$$d(x_m, x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \qquad (1.7)$$

$$\leq \left[\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}\right] d(x_0, x_1)$$

$$\leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1)$$

It follows that the sequence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Since X is f-orbitally complete, it follows that there exists a Cauchy subsequence $\{f^{n_i}(x)\}$ of $\{x_n\}$ in orbit $O(x, fx)$, $x \in X$, which converges to a point z in X. We now show that z is indeed, a fixed point of f. Suppose that $z \neq f(z)$. Since f is a p-contraction it follows that

$$d(f(x_{n_i}); f^2(x_{n_i})) \leq p(x_{n_i}) d(x_{n_i}, f(x_{n_i})) \qquad (1.8)$$

$$\leq \sup_{x \in X} p(f(x)) d(x_{n_i}, f(x_{n_i}))$$

Now taking the superior limit, we get

$$d(z, f(z)) = \lim_{i \to \infty} \sup d(f(x_{n_i}), f^2(x_{n_i})) \qquad (1.9)$$

$$\leq \sup_{x \in X} p(fx) \lim_{i \to \infty} \sup d(x_{n_i}, f(x_{n_i}))$$

$$= \sup_{x \in X} p(fx) d(z, f(z)) < d(z, f(z))$$

It is a contradiction. Thus z is a fixed point of f.

Example 2.1. Let $X = [-1, 1]$ equipped with usual metric d. Let $f : X \to X$ be a mapping

defined by $f(x) = \frac{4}{9}x$ for $x \geq 0$, $f(x) = 1$ for $x < 0$ $\qquad (1.10)$

Let

$$p : X \rightarrow [0,1] \qquad (1.11)$$

be a function defined by $p(x) = \frac{4}{9}\ \forall x \in X$ f is a fundamental p-contraction and it has also fixed point at $x = 0$.

$$d(fx, f^2x) \leq p(x)d(x, fx)$$

$$f(x) = \frac{4x}{9}, f\left(f\left(\frac{4x}{9}\right)\right) = f^2(x) = \frac{4 \cdot \frac{4x}{9}}{9} = \frac{16}{81}x \qquad (1.12)$$

$$d(fx, f^2x) = \frac{4}{9}x - \frac{16x}{81} = \frac{20x}{81} = \frac{4}{9} \cdot \frac{5x}{9} = p(x)d(x; fx)$$

$$= p(x). \qquad (1.13)$$

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